ON CARDINALITIES IN QUOTIENTS OF INVERSE LIMITS OF GROUPS

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ABSTRACT. Let λ be $\aleph_0$ or a strong limit of cofinality $\aleph_0$. Suppose that $\langle G_m, \pi_{m,n} : m \leq n < \omega \rangle$ and $\langle H_m, \pi^t_{m,n} : m \leq n < \omega \rangle$ are projective systems of groups of cardinality less than λ and suppose that for every $n < \omega$ there is a homorphism $\sigma : H_n \to G_n$ such that all the diagrams commute.

If for every $\mu < \lambda$ there exists $\langle f_i \in G_\omega : i < \mu \rangle$ such that $i \neq j \Longrightarrow f_i f_j^{-1} \notin \sigma_\omega(H_\omega)$ then there exists $\langle f_i \in G_\omega : i < 2^\lambda \rangle$ such that $i \neq j \Longrightarrow f_i f_j^{-1} \notin \sigma_\omega(H_\omega)$.

1. INTRODUCTION

The main result of this paper was motivated by our interest in the structure of the group $Ext_p(G, \mathbf{Z})$ for G abelian torsion free. For basic results about the structure of $Ext(G, \mathbf{Z})$ the reader is refered to sections 47 and 52 of Laszlo Fuchs book [Fu], however all we need is Definition 1.21 below. Since Shelah's proof of the independence of Whitehead's problem of ZFC (see [Sh 44]) much was done since that paper, for a summary see the introduction to [GrSh] and Chapter XII of Eklof & Mekler's book is dedicated ([EK]) to the structure of Ext.

In [GrSh] we have dealt with the cardinality of $Ext_p(G, \mathbf{Z})$. The main Theorem of [GrSh] states that for a strong limit λ of cofinality $\aleph_0$ for every torsion free G of cardinality λ either

$$|Ext_p(G, \mathbf{Z})| < \lambda \text{ or } |Ext_p(G, \mathbf{Z})| = 2^\lambda.$$

In section 2 of [GrSh] we indicated that the proof of the main theorem can be adapted to give a result concerning cardinalities of inverse systems of abelian groups subject to certain conditions (See Theorem 1.1 below). We did not include a proof there. Recently we

Date: February 8, 2008.

Partially supported by the United States-Israel Binational Science Foundation and by the National Science Fundation, NSF-DMS97-04477.

were asked to supply a complete proof to that theorem. Charles Megibben in a widely circulated preprint [Me] (which to our knowledge did not appear yet in print) even claimed that he proved a result that contradicts Theorem 1.1.

The aim of this paper is to present a complete proof of Theorem 1.1 below.

Notice that we do not make any assumptions on the groups, in particular the groups need not be commutative and can be even locally finite. See more on the subject in [Sh 664].

Theorem 1.1. *[The Main Theorem] Suppose λ is $\aleph_0$ or it is strong limit cardinal of cofinality $\aleph_0$.*

(1) *Let $\langle G_m, \pi_{m,n} : m \le n < \omega \rangle$ be an inverse system of groups of cardinality less than λ whose inverse limit is G_ω with $\pi_{n,\omega}$ such that $|G_n| < \lambda$. ($\pi_{m,n}$ is a homomorphism from G_m to $G_n, \alpha \le \beta \le \gamma \le \omega \Rightarrow \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\gamma}$ and $\pi_{\alpha,\alpha}$ is the identity).*
(2) *Let $\mathbf{I}$ be a finite index set. Suppose that for every $t \in \mathbf{I}$, $\langle H^t_m, \pi^t_{m,n} : m \le n < \omega \rangle$ is an inverse system of groups of cardinality less than λ and H^t_ω with $\pi^t_{n,\omega}$ be the corresponding inverse limit.*
(3) *Let for every $t \in \mathbf{I}, \sigma^t_n : H^t_n \to G_n$ be a homomorphism such that all diagrams commute (i.e. $\pi_{m,n} \circ \sigma^t_n = \sigma^t_m \circ \pi^t_{m,n}$ for $m \le n < \omega$), and let σ^t_ω be the induced homomorphism from H^t_ω into G_ω.*

Assume that for every $\mu < \lambda$ there is a sequence $\langle f_i \in G_\omega : i < \mu \rangle$ such that for $i \ne j$ and $t \in \mathbf{I} \Rightarrow f_i f_j^{-1} \notin$ $Rang(\sigma^t_\omega)$. Then there is $\langle f_i \in G_\omega : i < 2^\lambda \rangle$ such that $i \ne j$ and $t \in \mathbf{I} \Rightarrow f_i f_j^{-1} \notin$ $Rang(\sigma^t_\omega)$.

Notation 1.2. Since λ has cofinality $\aleph_0$ we can fix $\lambda_n < \lambda$ for $n < \omega$ such that $\lambda = \sum_{n<\omega} \lambda_n$, for all $n < \omega, \lambda_n$ is regular and $2^{\lambda_n} < \lambda_{n+1} < \lambda$ and $|G_n| + \sum_{t \in \mathbf{I}} |H^t_n| \le \lambda_n$.

Denote by $e_{G_\alpha}, e_{H^t_\alpha}$ the unit elements. Without loss of generality the groups are pairwise disjoint.

Definition 1.3. (1) For $\alpha \le \omega$ let $H_\alpha = \prod_{t \in \mathbf{I}} H^t_\alpha$ and $H_{<\alpha} = \prod_{\beta<\alpha} H_\beta$, $H_{\le\alpha} = \prod_{\beta\le\alpha} H_\beta$.
(2) For $\bar{g} \in H_\alpha$ let $\mathrm{lev}(\bar{g}) = \alpha$, for $g \in H^t_\alpha$ let $\mathrm{lev}(g) = \alpha$ (without loss of generality this is well defined).

(3) For $\alpha \leq \beta \leq \omega, g \in H^t_\beta$ let $g \restriction H^t_\alpha = \pi^t_{\alpha,\beta}(g)$ and we say $g \restriction H^t_\alpha$ *is below* g and g *is above* $g \restriction H^t_\alpha$ or *extend* $g \restriction H^t_\alpha$.
(4) For $\alpha \leq \beta \leq \omega, f \in G_\beta$ let $f \restriction G_\alpha = \pi_{\alpha,\beta}(f)$.

We will now introduce the rank function used in the proof of Theorem 1.1, it is a measure for the possibility to extend functions in Lemma 1.7 we show that it is an ultrametric valuation.

Definition 1.4. (1) For $g \in H^t_n, f \in G_\omega$ we say that (g, f) *is a nice* t*-pair* if $\sigma^t_n(g) = f \restriction G_n$.
(2) Define a ranking function $\mathrm{rk}_t(g, f)$ for any nice t-pair. First by induction on the ordinal α (we can fix $f \in G_\omega$), we define when $\mathrm{rk}_t(g, f) \geq \alpha$ simultaneously for all $n < \omega$ and every $g \in H^t_n$
  (a) $\mathrm{rk}_t(g, f) \geq 0$ iff (g, f) is a nice t-pair
  (b) $\mathrm{rk}_t(g, f) \geq \delta$ for a limit ordinal δ iff for every $\beta < \delta$ we have $\mathrm{rk}_t(g, f) \geq \beta$
  (c) $\mathrm{rk}_t(g, f) \geq \beta + 1$ iff (g, f) is a nice t-pair, and letting $n = \mathrm{lev}(g)$ there exists $g' \in H^t_{n+1}$ extending g such that $\mathrm{rk}_t(g', f) \geq \beta$
  (d) $\mathrm{rk}_t(g, f) \geq -1$.
(3) For α an ordinal or -1 (stipulating $-1 < \alpha < \infty$ for any ordinal α) we have $\mathrm{rk}_t(g, f) = \alpha$ iff $\mathrm{rk}_t(g, f) \geq \alpha$ and it is false that $\mathrm{rk}_t(g, f) \geq \alpha + 1$.
(4) $\mathrm{rk}_t(g, f) = \infty$ iff for every ordinal α we have $\mathrm{rk}_t(g, f) \geq \alpha$.

The following two claims give the principal properties of $\mathrm{rk}_t(g, f)$.

Claim 1.5. *Let (g, f) be a nice t-pair.*

(1) *The following statements are equivalent:*
  (a) $rk_t(g, f) = \infty$
  (b) *there exists $g' \in H^t_\omega$ extending g such that $\sigma^t_\omega(g') = f$.*
(2) *If $rk_t(g, f) < \infty$, then $rk_t(g, f) < \lambda^+$.*
(3) *If g' is a proper extension of g and (g', f) is also a nice t-pair then*
  (a) $rk_t(g', f) \leq rk_t(g, f)$ *and*
  (b) *if $0 \leq rk_t(g, f) < \infty$ then the inequality is strict.*

Proof. (1) $(a) \Rightarrow (b)$: Let n be such that $g \in H^t_n$. It is enough to define $g_k \in H^t_k$ for $k < \omega, k \geq n$ such that
  (a) $g_n = g$
  (b) g_k is below g_{k+1} that is $\pi^t_{k,k+1}(g_{k+1}) = g_k$ and
  (c) $\mathrm{rk}_t(g_{k+1}, f) = \infty$:

Let $g' := \varinjlim g_k$ it is as required. The definition is by induction on $k \geq n$. For $k = n$ let $g_0 = g$. For $k \geq n$, suppose g_k is defined. By (iii) we have $\mathrm{rk}_t(g_k, f) = \infty$, hence there exists $g^* \in H^t_{k+1}$ extending g_k such that $\mathrm{rk}_t(g^*, f) = \infty$, and let $g_{k+1} := g^*$.

$(b) \Rightarrow (a)$: Since g is below g', it is enough to prove by induction on α that for every $k \geq n$ when $g_k := g' \restriction H^t_k$ we have that $\mathrm{rk}_t(g, f) \geq \alpha$.

For $\alpha = 0$, since $\sigma^t_\omega(g') = f \restriction G_n$ clearly for every k we have $\sigma^t_k(g_k) = f \restriction G_k$ so (g_k, f) is a nice t-pair.

For limit α, by the induction hypothesis for every $\beta < \alpha$ and every k we have $\mathrm{rk}_t(g_k, f) \geq \beta$, hence by Definition 1.4(2)(b), $\mathrm{rk}_t(g_k, f) \geq \alpha$.

For $\alpha = \beta + 1$, by the induction hypothesis for every $k \geq n$ we have $\mathrm{rk}_t(g_k, f) \geq \beta$. Let $k_0 \geq n$ be given. Since g_{k_0} is below g_{k_0+1} and $\mathrm{rk}_t(g_{k_0+1}, f) \geq \beta$, Definition 1.4(2)(c) implies that $\mathrm{rk}_t(g_{k_0}, f) \geq \beta + 1$; i.e. for every $k \geq n$ we have $\mathrm{rk}_t(g_k, f) \geq \alpha$. So we are done.

(2) Let $g \in H^t_n$ and $f \in G_\omega$ be given. It is enough to prove that if $\mathrm{rk}_t(g, f) \geq \lambda^+$ then $\mathrm{rk}_t(g, f) = \infty$. Using part (1) it is enough to find $g' \in H^t_\omega$ such that g is below g' and $f = \sigma^t_\omega(g')$.

We define by induction on $k < \omega, g_k \in H^t_{n+k}$ such that g_k is below g_{k+1} and $\mathrm{rk}_t(g_k, f) \geq \lambda^+$. For $k = 0$ let $g_k = g$. For $k + 1$, for every $\alpha < \lambda^+$, as $\mathrm{rk}_t(g_k, f) > \alpha$ by 1.4(2)(c) there is $g_{k,\alpha} \in G_{n+k+1}$ extending g_k such that $\mathrm{rk}_t(g_{k,\alpha}, f) \geq \alpha$. But the number of possible $g_{k,\alpha}$ is $\leq |H^t_{n+k+1}| \leq 2^{\lambda_{n+k+1}} < \lambda^+$ hence there are a function g and a set $S \subseteq \lambda^+$ of cardinality λ^+ such that $\alpha \in S \Rightarrow g_{k,\alpha} = g$. Now take $g_{k+1} = g$.

(3) Immediate.

$\square$

Lemma 1.6. (1) *Let (g, f) be a nice t-pair. Then we have $rk(g, f) \leq rk(g^{-1}, f^{-1})$.*

(2) *For every nice t-pair (g, f) we have $rk(g, f) = rk(g^{-1}, f^{-1})$.*

Proof. (1) By induction on α prove that $\mathrm{rk}(g, f) \geq \alpha \Rightarrow \mathrm{rk}(g^{-1}, f^{-1}) \geq \alpha$ (see more details in Lemma 1.7).

(2) Apply part (1) twice.

$\square$

In the following lemma we show that the rank is indeed ultrametric (ordinal valued).

Lemma 1.7. *Let $n < \omega$ be fixed, and let $(g_1, f_1), (g_2, f_2)$ be nice t-pairs with $g_\ell \in H^t_n (\ell = 1, 2)$.*

(1) *If (g_1, f_1) and (g_2, f_2) are t-nice pairs, then $(g_1 g_2, f_1 f_2)$ is a nice pair and $rk_t(g_1 g_2, f_1 f_2) \geq \; Min\{rk_t(g_\ell, f_\ell) : \ell = 1, 2\}$.*
(2) *Let $n, (f_1, g_1)$ and (f_2, g_2) be as above. If $rk_t(g_1, f_1) \neq \; rk_t(g_2, f_2)$, then $rk_t(g_1 g_2, f_1 f_2) = \; Min\{rk_t(g_\ell, f_\ell) : \ell = 1, 2\}$.*

Proof. (1) It is easy to show that the pair is t-nice. We show by induction on α simultaneously for all $n < \omega$ and every $g_1, g_2 \in H^t_n$ that $\mathrm{Min}\{\mathrm{rk}(g_\ell, f_\ell) : \ell = 1, 2\} \geq \alpha$ implies that $\mathrm{rk}(g_1 g_2, f_1 f_2) \geq \alpha$.

When $\alpha = 0$ or α is a limit ordinal this is easy. Suppose $\alpha = \beta + 1$ and that $\mathrm{rk}(g_\ell, f_\ell) \geq \beta + 1$; by the definition of rank for $\ell = 1, 2$ there exists $g'_\ell \in H^t_{n+1}$ extending g_ℓ such that (g'_ℓ, f_ℓ) is a nice pair and $\mathrm{rk}_t(g'_\ell, f_\ell) \geq \beta$. By the induction assumption $\mathrm{rk}_t(g'_1 g'_2, f_1 f_2) \geq \beta$. Hence $g'_1 g'_2$ is as required in the definition of $\mathrm{rk}_t(g_1 g_2, f_1 f_2) \geq \beta + 1$.

(2) Suppose without loss of generality that $\mathrm{rk}(g_1, f_1) < \; \mathrm{rk}(g_2, f_2)$, let $\alpha_1 = \; \mathrm{rk}(g_1, f_1)$ and let $\alpha_2 = \; \mathrm{rk}_t(g_2, f_2)$. By part (1), $\mathrm{rk}_t(g_1 g_2, f_1 f_2) \geq \alpha_1$, by Proposition 1.6, $\mathrm{rk}_t(g_2^{-1}, f_2^{-1}) = \alpha_2 > \alpha_1$. So we have

$$\begin{aligned} \alpha_1 = \; & \mathrm{rk}_t(g_1, f_1) = \; \mathrm{rk}_t(g_1 g_2 g_2^{-1}, f_1 f_2 f_2^{-1}) \\ \geq \; & \mathrm{Min}\{\mathrm{rk}_t(g_1 g_2, f_1 f_2), \mathrm{rk}_t(g_2^{-1}, f_2^{-1})\} \\ = \; & \mathrm{rk}_t(g_1 g_2, f_1 f_2) \geq \alpha_1. \end{aligned}$$

Hence the conclusion follows. ☐

Definition 1.8. (1) Let $\mu < \lambda$ and let $\bar{\alpha} = \langle \alpha_t : t \in \mathbf{I} \rangle$ where α_t is an ordinal less or equal to λ^+. We say that $\bar{f} = \langle f_i : i < \mu \rangle$ *μ-exemplifies* $\bar{\alpha} \in \Gamma_n$ (or $\bar{f}$ *is a μ-witness for* $\bar{\alpha} \in \Gamma_n$) iff

(a) $f_i \in G_\omega$ and $f_i \restriction G_n = e_{G_n}$
(b) for $i \neq j$ and $t \in \mathbf{I}$ we have $\mathrm{rk}_t(e_{H^t_n}, f_i f_j^{-1}) < \alpha_t$ (possibly is -1).

(2) Let

$$\Gamma_n = \Bigg\{ \bar{\alpha} \; : \; \bar{\alpha} = \langle \alpha_t : t \in \mathbf{I} \rangle, \alpha_t \text{ an ordinal } \leq \lambda^+, \\ \text{and for every } \mu < \lambda \text{ there is a sequence } \langle f_i : i < \mu \rangle \\ \text{which } \mu\text{-exemplifies } \bar{\alpha} \in \Gamma_n \Bigg\}.$$

(3) $\Delta_n = \{\bar\alpha \in \Gamma_n : \text{for no } \bar\beta \text{ we have } \bar\beta \in \Gamma_n, \bar\beta \le \bar\alpha \text{ (i.e. } \bigwedge_{t\in \mathbf{J}_n} \beta_t \le \alpha_t) \text{ and } \bar\beta \ne \bar\alpha\}$.

Claim 1.9. (1) *Γ_n is not empty.*

(2) *Δ_n is not empty in fact $(\forall \bar\alpha \in \Gamma_n)(\exists \bar\beta \in \Delta_n)(\bar\beta \le \bar\alpha)$.*

Proof. (1) Let $\alpha^*_t = \sup\{\mathrm{rk}_t(g,f)+1 : g \in H^t_n, f \in G^\omega \text{ and } \mathrm{rk}_t(g,f) < \infty\}$, by 1.5(2), this is a supremum on a set of ordinals $< \lambda^+$ (as $-1+1=0$) hence is an ordinal $\le \lambda^+$. So $\langle \alpha^*_t : t \in \mathbf{I}\rangle$ is as required.

(2) If not, then choose by induction on $\ell < \omega$ a sequence $\bar\beta^\ell \in \Gamma_n$ such that $\bar\beta^0 = \bar\alpha, \bar\beta^{\ell+1} \le \bar\beta^\ell, \bar\beta^{\ell+1} \ne \beta^\ell$. So for each $t \in \mathbf{I}$, the sequence $\langle \beta^\ell_t : \ell < \omega\rangle$ is a non-increasing sequence of ordinals hence is eventually constant, say for some $\ell_t < \omega$ we have $\ell \in [\ell_t, \omega) \Rightarrow \beta^\ell_t = \beta^{\ell_t}_t$, so as $\mathbf{I}$ is finite, $\ell(*) = \max\{\ell_t : t \in \mathbf{I}\} < \omega$, so $\bar\beta^{\ell(*)} = \bar\beta^{\ell(*)+1}$, a contradiction.

$\square$

Claim 1.10. (1) *If $\mu \le \mu'$ and $\langle f_i : i < \mu'\rangle, \mu'$-exemplify $\bar\alpha \in \Gamma_n$ and $h : \mu \to \mu'$ is one to one, then $\langle f_{h(i)} : i < \mu\rangle, \mu$-exemplifies $\bar\alpha \in \Gamma_n$.*

(2) *If $\langle f_i : i < \mu\rangle, \mu$-exemplify $\bar\alpha \in \Gamma_n$ and $f_i \restriction G_{n+1} = f$ for $i < \mu$, then $\langle f_i f_0^{-1} : i < \mu\rangle, \mu$-exemplify $\bar\alpha \in \Gamma_{n+1}$.*

(3) *If $\bar\alpha \in \Gamma_n$, then $\bar\alpha \in \Gamma_{n+1}$.*

(4) *If $\bar\alpha \in \Delta_n$, then some $\bar\beta \le \bar\alpha$ belongs to Δ_{n+1}.*

(5) *For some $n < \omega$ there is $\bar\alpha \in \bigcap_{m\ge n} \Delta_n$.*

(6) *In clause (b) of Definition 1.8(1) it suffices to deal with $i < j$.*

Proof. (1) Trivial.

(2) Clearly.

Clause (a):

$(f_i \circ f_0^{-1}) \restriction G_{n+1} = \sigma^\omega_{n+1}(f_i f_0^{-1}) = (\sigma^\omega_{n+1}(f_i))(\sigma^\omega_{n+1}(f_0))^{-1} = f f^{-1} = e_{G_{n+1}}$.

Clause (b):

For $i \ne j$ and $t \in \mathbf{I}$, note that

$$(f_i f_0^{-1})(f_j f_0^{-1}) = f_i f_0^{-1} f_0 f_j^{-1} = f_i f_j^{-1}$$

so we can use the assumption.

(3) So let $\mu < \lambda$ and we should find a μ-witness for $\bar\alpha \in \Gamma_{n+1}$. We can choose μ' such that $\mu \times |G_{n+1}| < \mu' < \lambda$. As $\bar\alpha \in \Gamma_n$, clearly there is a μ'-witness $\langle f_i : i < \mu'\rangle$ for it. Now the number of possible $f_i \restriction$

G_{n+1} is $\leq |G_{n+1}|$ (really) even $\leq |\mathrm{Rang}(\pi_{n+1,\omega}) \cap \ \mathrm{Ker}(\pi_{n,n+1})|)$ hence for some $f \in G_{n+1}$ and $Y \subseteq \mu'$ we have: $|Y| \geq \mu$ and $i \in Y \Rightarrow f_i \restriction G_{n+1} = f$. By renaming $\{i : i < \mu\} \subseteq Y$, now $\langle f_i f_0^{-1} : i < \mu \rangle$ is a μ-witness by part (1).

(4) Follows by 1.10(2) and 1.9(2).
(5) By 1.10(3) by the well foundedness of the ordinals (as in the proof of 1.9(2),(8).
(6) Because for $i < j, (f_j f_i^{-1})^{-1} = (f_i f_j^{-1})$ and 1.6(2).

$\square$

Convention 1.11. By renaming and 1.10(4), without loss of generality $\bar{\alpha}^* \in \Delta_n$ for every n.

Claim 1.12. *Each $\alpha^*_t (t \in \mathbf{I})$ is a non-successor ordinal (i.e. limit or zero).*

Proof. Fix $n < \omega$.

Assume $s \in \mathbf{I}$ is a counterexample. So $\alpha^*_s = \beta^* + 1, \beta^* \geq 0$. Let $\bar{\beta} = \langle \beta_t : t \in \mathbf{I} \rangle$ be defined as follows: β_t is α_t if $t \neq s$ and is β^* if $t = s$. We shall prove that $\bar{\beta} \in \Gamma_{n+1}$ thus getting a contradiction. So let $\mu < \lambda$ and we shall find a μ-witness for $\bar{\beta} \in \Gamma_{n+1}$. Let μ' be such that $\mu|G_{n+1}| < \mu' < \lambda$. As $\bar{\alpha}^* \in \Gamma_n$ (see 1.11) there is a μ'-witness $\langle f_i : i < \mu' \rangle$ for $\bar{\alpha}^* \in \Gamma_n$, as earlier without loss of generality $i < \mu \Rightarrow f_i \restriction G_{n+1} = f$ for some f. We shall prove that $\langle f_i f_0^{-1} : i < \mu \rangle$ is a μ-witness for $\bar{\beta} \in \Gamma_{n+1}$. Let $f'_i = f_i f_0^{-1}$ for $i < \mu$.

Clause (a):

$f' \restriction G_{n+1} = (f_0 f_0^{-1}) \restriction G_{n+1} = e_{G_{n+1}}$ because $f_i \restriction G_{n+1} = f_0 \restriction G_{n+1}$.

Clause (b):

Let $i \neq j < \mu$. If $t \in \mathbf{I} \backslash \{s\}$ then $\mathrm{rk}_t(e_{G_{n+1}}, f'_i (f'_j)^{-1}) = \ \mathrm{rk}_t(e_{G_{n+1}}, f_i f_j^{-1}) \leq \ \mathrm{rk}_t(e_{G_n}, f_i f_j^{-1}) \leq \alpha^*_t = \beta_t$. (Why? By group theory, by 1.5(3)(α), by choice of $\bar{f}$, by choice of β_t, respectively).

If $t = s$, then $\mathrm{rk}_t(e_{G_n}, f_i f_j^{-1}) < \ \mathrm{rk}_t(e_{G_{n+1}}, f_i f_j^{-1})$ by 1.5(3)(β), and proceed as above. $\square$

Notation 1.13. For $\alpha \le \omega$ let $T_\alpha := \prod_{k<\alpha} \lambda_k, T := \prod_{n<\omega} T_n$ (note: treeness used).

Claim 1.14. *There are for $n < \omega$, a sequence $\langle f_{n,i} : i < \lambda_n \rangle$ and an ordinal $\gamma_n^t < \alpha_t^*$ (α_t^* is the ordinal from 1.11) such that*

(1) $f_{n,i} \in G_\omega, f_{n,i} \restriction G_{n+1} = e_{G_{n+1}}$ *for all* $i < \lambda_n$;
(2) *for each* $t \in \mathbf{I}$ *for every* $h \in H_n^t$ *and* $i < j < \lambda_n$ *we have:* $rk_t(h, f_{n,i} f_{n,j}^{-1}) \le \gamma_n^t$;
(3) $rk_t(e_{H_n^t}, f_{n,i} f_{n,j}^{-1}) \ge \gamma_{n-1}^t$ *for* $i < j < \lambda_n$ *and* $\gamma_{n-1}^t \ge 0 \Rightarrow \; rk_t(e_{H_n^t}, f_{n,i} f_{n,j}^{-1}) > \gamma_{n-1}^t$
(4) $\gamma_{n-1}^t < \gamma_n^t$ *if* $\alpha_t^* > 0$ *and* $\gamma_n^t = -1$ *if* $\alpha_t^* = 0$.

We delay the rest of proof for a while.

Convention 1.15. Let $\gamma_n^t, g_{n,i}\,(n < \omega, i < \lambda_n)$ be as in 1.14.

Definition 1.16. We set $f_\eta = g_{n-1,\eta(n-1)} g_{n-2,\eta(n-2)} \cdots g_{0,\eta(0)}$ for $\eta \in T_n$. Then define f_η for $\eta \in T_\omega$ as follows: f_η is the element of G^ω satisfying $f_\eta \restriction G_n = f_{\eta\restriction n}$. It is well defined by:

Fact 1.17. (1) *For $\eta \in T_\omega$ and $m \le n < \omega$ we have*

$$f_{\eta\restriction n} \restriction G_{n+1} = f_{\eta\restriction m} \restriction G_{n+1}.$$

(2) *For $\eta \in T_\omega$ we have $f_\eta \in G_\omega$ is well defined (as the inverse limit of $\langle f_{\eta\restriction n} \restriction G_n : n < \omega \rangle$, so $n < \omega \to f_\eta \restriction G_n = f_{\eta\restriction n}$.*

Proof. (1) As $\pi_{n,\omega}$ is a homomorphism it is enough to prove $(f_{\eta\restriction n}(f_{\eta\restriction m})^{-1}) \restriction G_{n+1} = e_{G_{n+1}}$, hence it is enough to prove $n \le k < \omega \Rightarrow (f_{\eta\restriction k} f_{\eta\restriction(k+1)}^{-1}) \restriction G_{n+1} = e_{G_{n+1}}$ which follows from $k < \omega \Rightarrow f_{\eta\restriction k} f_{\eta\restriction(k+1)}^{-1} \restriction G_{k+1} = e_{G_{k+1}}$, which means $f_{k,\eta(k)} \restriction G_{k+1} = e_{G_{k+1}}$ which holds by clause (a) of 1.11.
(2) Follows by part (1) and G_ω being an inverse limit.

$\square$

Proposition 1.18. *Let $\eta, \nu \in T_\omega$. If $\eta \ne \nu$ and $t \in \mathbf{I}$, then $f_\eta f_\nu^{-1} \notin \sigma_\omega^t(H_\omega^t)$.*

Proof. Suppose for the sake of contradiction that for some $g \in H^t_\omega$ we have $\sigma^t_\omega(g) = f_\eta f_\nu^{-1}$.

Let k be minimal such that $\eta \restriction k = \nu \restriction k, \eta(k) \neq \nu(k)$, without loss of generality $\eta(k) < \nu(k)$. For $\ell \geq k$ let ξ^ℓ be $\mathrm{rk}_t(g \restriction H^t_\ell, f_{\eta\restriction(\ell+1)} f^{-1}_{\nu\restriction(\ell+1)})$. We will reach a contradiction by showing that $\ell \geq k \Rightarrow 0 \leq \xi^\ell \leq \gamma^t_k$ and $\ell > k \Rightarrow \xi^{\ell+1} < \xi^\ell$.

Note

$(*)_1$ if $\ell \leq \alpha \leq \omega$, then $\mathrm{rk}_t(g \restriction H^t_\ell, f_{\eta\restriction\alpha} f^{-1}_{\nu\restriction\alpha}) \geq 0$ as $\sigma^t_\ell(g \restriction H^t_\ell) = \sigma^t(g) \restriction G^t_\ell = (f_\eta f_\nu^{-1}) \restriction F^t_\ell$ and 1.17.

For $\ell = k$, we show that $\xi^k \leq \gamma^t_k$. Let $i = \eta[k], j = \nu[k]$. By the choice of $k, i \neq j$. In this case $f_{\eta\restriction(\ell+1)} f^{-1}_{\nu\restriction(\ell+1)} = f_{k,\eta(k)} f^{-1}_{k,\nu(k)}$ by the minimality of k and, of course, $f_{k,\eta(k)} f^{-1}_{k,\nu(k)} = f_{k,i} f^{-1}_{k,j}$, hence $\xi^k = \mathrm{rk}_t(g \restriction H^h_k, f_{k,i} f^{-1}_{k,j}) \leq \gamma_k$ by clause (b) of 1.14. Note: if $\alpha^*_t = 0$, then $\gamma^t_m = -1$ for $m < \omega$ hence $\xi^k = -1$, but $(f_\eta f_\nu^{-1}) \restriction G_k = (f_{\eta\restriction(k+1)} f^{-1}_{\nu\restriction(k+1)}) \restriction G_k$ immediate contradiction. So assume $\alpha^*_t \geq 0$ hence $0 \leq \gamma^t_m < \gamma^t_{m+1}$.

Now we proceed inductively. We assume that $\xi^\ell \leq \xi^k$ and show that $\xi^{\ell+1} < \xi^\ell$. Let $i = \eta[\ell+1], j = \nu[\ell+1]$, and let $\zeta = \mathrm{rk}_t(g \restriction H^t_{\ell+1}, f_{\eta\restriction(\ell+1)} f^{-1}_{\nu\restriction(\ell+1)})$. Observe:

$(*)_2$ $\zeta < \mathrm{rk}_t(g \restriction H^t_\ell, f_{\eta\restriction(\ell+1)} f^{-1}_{\nu\restriction(\ell+1)}) = \xi^\ell$ [why? by 1.5(3) and $(*)_1$ above.]

So

$(*)_3$
$$\begin{aligned}\xi^{\ell+1} &= \mathrm{rk}_t(g \restriction H^t_{\ell+1}, f_{\eta\restriction(\ell+2)} f^{-1}_{\nu\restriction(\ell+2)})\\ &= \mathrm{rk}_t(g \restriction H^t_{\ell+1}, f_{\ell+1,\eta(\ell+1)}(f_{\eta\restriction(\ell+1)} f^{-1}_{\nu\restriction(\ell+1)}) f_{\ell+1,\nu(\ell+1)})\\ &= \mathrm{rk}_t(e_{H^t_{\ell+1}}(g \restriction H^t_{\ell+1}) e_{H^t_{\ell+1}}, f_{\ell+1,\eta(\ell+1)}(f_{\eta\restriction(\ell+1)} f^{-1}_{\nu\restriction(\ell+1)}) f_{\ell+1,\nu(\ell+1)}).\end{aligned}$$

Now:

$(*)_4$ $\mathrm{rk}_t(e_{H^t_{\ell+1}}, f_{\ell+1,\eta(\ell+1)}) > \gamma^t_\ell$ (why? by clause (c) of 1.14)

$(*)_5$ $\mathrm{rk}_t(g \restriction H^t_{\ell+1}, f_{\eta\restriction(\ell+1)} f^{-1}_{\nu\restriction(\ell+1)}) = \xi^\ell \leq \xi^k \leq \gamma^t_k \leq \gamma^t_\ell$
(why? the equality by the definition of ξ^ℓ, the first inequality by the induction hypothesis and the second inequality was proved above (for $\ell = k$), the last inequality by 1.14 clause (d)

$(*)_6$ $\mathrm{rk}_t(e_{H^t_{\ell+1}}, g_{\ell+1,\nu(\ell+1)}) > \gamma^t_\ell$ (why? by clause (c) of 1.14).

Hence by 1.5(3)

$(*)_7$ $\mathrm{rk}_t(e_{H^t_{\ell+1}}(g \restriction H^t_{\ell+1})e_{H^t_{\ell+1}}, f_{\ell+1,\eta(\ell)}(f_{\eta\restriction(\ell+1)}f^{-1}_{\nu\restriction(\ell+1)})f_{\ell+1,\nu(\ell+1)})$
$= \mathrm{rk}(g \restriction G_{\ell+1}, f_{\eta\restriction(\ell+1)}f^{-1}_{\nu\restriction(\ell+1)})$.

Together we get the induction demand for $\ell + 1$. $\square$

Before proving 1.14 and finishing we prove

Claim 1.19. *Assume $-1 \leq \beta_t < \alpha^*_t$ for $t \in \mathbf{I}$ and $n < \omega$ and $\mu < \lambda$. Then we can find $\langle f_i : i < \mu\rangle$ such that*

(1) $f_i \in G_\omega$ *and* $f_i \restriction G_{n+1} = e_{G_{n+1}}$
(2) $t \in \mathbf{I}$ *and* $i \neq j \Rightarrow$ $rk_t(e_{H^t_n}, f_i f_j^{-1}) \in [\beta^t, \alpha^*_t)$
(3) $t \in \mathbf{I}$ *and* $i < \mu =$ $rk_t(e_{H^t_n}, f_i) \in [\beta_t, \alpha^*_t)$.

Proof. For each $s \in \mathbf{I}$ we define $\bar\beta^s = \langle \beta^s_t : t \in I\rangle$ by:

$$\beta^s_t = \begin{cases} \alpha_t & \text{if } t \neq s \\ \beta_t & \text{if } t = s \end{cases}$$

So $\bar\beta^s \leq \bar\alpha^*, \bar\beta^s \neq \bar\alpha^*$, so as $\bar\alpha^* \in \times_{n<\omega}\Delta_m$ necessarily $\bar\beta^s \notin \Gamma_n$, hence for some $\mu^s < \lambda$ there is no μ^s-witness for $\bar\beta^s$ and n (check the definition of Γ_n).

Let $\mu_1 < \lambda$ be $> \mu +$ $\max\{\mu^s : s \in \mathbf{I}\}$.

Let $\chi < \lambda$ be large enough (so that it will be possible to use the finite Ramsey theorem when $\lambda = \aleph_0$ and when $\lambda > \aleph_0$ the Erdös Rado theorem we require that $\chi \rightarrow (\mu_1)^2_\theta$ where $\theta = 2^{\sum_t |H^t_n|}$).

Let $\langle f_i : i < \chi\rangle$ be a χ-witness $\bar\alpha \in \Gamma_n$ and even $\bar\alpha \in \Gamma_{n+1}$. For each $t \in \mathbf{I}, h \in H^t_n$ define the two place function $F_{t,h}$ from $[\chi]^2$ to $\{0,1\}$ for $i < j < \chi$ let

$$F_{t,h}\{i,j\} := \begin{cases} 0 & \text{if} \quad \mathrm{rk}_t(h, f_i f_j^{-1}) < \beta_t \\ 1 & \text{Otherwise.} \end{cases}$$

Define the two-place function F from $[\chi]^2$: For $i < j < \chi$ let $F\{i,j\} = \langle F_{t,h}(i,j) : t \in \mathbf{I}, h \in H^t_n\rangle$.

Clearly $|\mathrm{Rang}(F)| \leq 2^{\sum_t |H^t_n|}$.

Hence an application of one of the above partition theorems provides us with a set $Y \subseteq \chi, |Y| = \mu_1$ such that $F \restriction [Y]^2$ is constant. Without loss of generality $Y = \mu_1$.

For each $s \in \mathbf{I}$, clearly $\langle f_i f_0^{-1} : i < \mu^s \rangle$ is not a μ^s-witness for $\bar{\beta}^s$, but the only thing that may go wrong is the inequality, $i < j < \mu^s \Rightarrow \operatorname{rk}_s(e_{H_n^s}, f_i f_j^{-1}) < \beta_s$, so for some $i < j < \mu^s$ we have that $\operatorname{rk}_s(e_{H_n^s}, f_i f_j^{-1}) \geq \beta_s$ holds, hence

$(*)$ $s \in \mathbf{I}$ and $i < j < \mu_1 \Rightarrow \operatorname{rk}_s(e_{H_n^s}, f_i f_j^{-1}) \geq \beta_s$.

This means clause (b) holds and clause (a) by definition of $\langle f_i : i < \chi \rangle$ is a χ-witness for $\bar{\alpha} \in \Gamma_n$. Clause (c) follows. So $\langle f_i : i < \mu \rangle$ is as required. $\square$

Proof. of 1.14

Stipulate γ_{-1}^t: if $\alpha_t^* > 0$ it is 0, otherwise is it -1. Assume $n < \omega$ and $\langle \gamma_{n-1}^t : t \in \mathbf{I} \rangle$ is well defined, $\gamma_{n-1}^t < \alpha_t^*$. Let $\gamma_n^{t,*}$ be: $\gamma_{n-1}^t + 1$ if α_t^* is a limit ordinal and $\gamma_{n-1}^t = -1$ otherwise (i.e. $\alpha_t^* = 0$, see 1.12). Note that to construct the family $\{f_{n,i} : i < \lambda_n\}$ we will combine Claim 1.19 with a second application of the Erdös Rado Theorem.

Let $\theta = (2^{|H_n^t| \times |H_n^t|}) \times |\mathbf{I}|$ and $\chi < \lambda$ be such that $\chi \to (\lambda_n + 2)_\theta^3$ (exists by Ramsey theorem if $\lambda = \aleph_0$ and by Erdös Rado theorem if $\lambda > \aleph_0$). Apply Claim 1.19 to get a family $\{f_i : i < \chi\}$ satisfying:

(1) $f_i \restriction G_{n+1} = e_{G_{n+1}}$,
(2) for $i \neq j$ and $t \in \mathbf{I}$, we have $\gamma_{n-1}^{t,*} \leq \operatorname{rk}_t(e_{H_n^t}, f_i f_j^{-1}) < \alpha_t^*$.

For $t \in \mathbf{I}, \bar{g} = \langle g_1, g_2 \rangle, g_1, g_2 \in H_n^t$ such that $\sigma_n^t(g) = e_{G_n}$ define a coloring $F_{t,\bar{g}}$ of $[I]^3$ by two colors according to the following scheme: for $\varepsilon < \zeta < \xi < \chi$, let

$$F_{t,g}\{\varepsilon, \zeta, \xi\} := \begin{cases} \textit{red} & \text{if} \quad \operatorname{rk}_t(g_1, f_{i_\varepsilon} f_\zeta^{-1}) \leq \operatorname{rk}_t(g_2, f_\zeta f_\xi^{-1}); \\ \textit{green} & \text{if} \quad \operatorname{rk}_t(g_1, f_{i_\varepsilon} f_\zeta^{-1}) > \operatorname{rk}_t(g_2, f_\zeta f_\xi^{-1}) \end{cases} .$$

By the Ramsey theorem (if $\lambda = \aleph_0$) or Erdös Rado Theorem if $\lambda > \aleph_0$ there is a set $J \subseteq \chi$, $\operatorname{otp}(J) = \lambda_n + 2$ such that each coloring is constant on $[J]^3$. Let the value of $F_{t,\bar{g}}$ on $[J]^3$ be denoted $c_{t,\bar{g}}$. Observe that $c_{t,\bar{g}}$ is never *green* as this would produce a descending ω-sequence of ordinals as if $\varepsilon_\ell \in J, \varepsilon_\ell < \varepsilon_{\ell+1}$ for $\ell < \omega$, then $\operatorname{rk}_t(g, f_{\varepsilon_\ell} f_{\varepsilon_{\ell+1}}^{-1}) > \operatorname{rk}_t(g, f_{\varepsilon_{\ell+1}} f_{\varepsilon_{\ell+2}}^{-1})$, so $\langle \operatorname{rk}_t(g, f_{\varepsilon_{2\ell}} f_{\varepsilon_{2\ell+1}}^{-1}) : \ell < \omega \rangle$ is strictly decreasing.

Let $\varepsilon(*) = \mathrm{Min}(J)$ and $J_0 = \{\varepsilon \in J : \mathrm{otp}(\varepsilon \cap J) < \lambda\}$ and α is the λ_n-th member of J, β the $(\lambda_n + 1)$-th member of J and let $\gamma_n^t = \mathrm{rk}_t(e_{H_n^t}, f_\alpha f_\beta^{-1})$, by clause (b) above $\gamma_n^{t,*} \leq \gamma_n^t < \alpha_t^*$ so $\alpha_t^* = 0 \Rightarrow \gamma_n^t = -1$ and $\alpha_t^* > 0 \Rightarrow \gamma_{n-1}^t < \gamma_n^t$.

We claim that $\{f_i f_{\varepsilon(*)}^{-1} : i \in J_0\}$ (remember $J_0 \subseteq J, |J_0| = \lambda_n$) provides a set that can play the role of $\{f_{n,i} : i < \lambda_n\}$. We note

$(*)_1$ $\mathrm{rk}_t(g, f_\varepsilon f_\zeta^{-1}) \leq \gamma_t^n$ for $\varepsilon < \zeta$ in J_0 [why? clearly $\alpha < \beta < \varepsilon < \zeta$ are in J hence by the choice of J we have $\mathrm{rk}_t(g, f_\varepsilon f_\zeta^{-1}) \leq \mathrm{rk}_t(g, f_\zeta f_\alpha^{-1}) \leq \mathrm{rk}_t(g, f_\alpha f_\beta^{-1}) = \gamma_n^t$].

Now clauses (1), (4) of 1.14 holds by clause (1) above, clause (3) of 1.14 holds by $(*)_1$ and clause (4) of 1.14 holds by the choice of the γ_t^*. We are left with clause (2). Let $h \in H_n^t$, as above clearly for $\Upsilon < \xi < \zeta < \xi$ in J we have $\mathrm{rk}_t(h, f_\varepsilon f_\zeta^{-1}) \leq \mathrm{rk}_t(h, f_\zeta f_\xi^{-1})$. Hence for $\Upsilon\varepsilon < \zeta < \xi$ in J_0 we have

$$\begin{aligned}
\gamma_t^n \geq\ & \mathrm{rk}_t(e_{H_n^t}, f_\varepsilon f_\zeta^{-1}) \\
=\ & \mathrm{rk}_t(h^{-1}, (f_\varepsilon f_\xi^{-1})(f_\zeta f_\xi^{-1})^{-1}) \\
\geq\ & \mathrm{Min}\{\mathrm{rk}_t(h, f_\varepsilon f_\xi^{-1}),\ \mathrm{rk}_t(h^{-1}, (f_\zeta f_\xi^{-1})^{-1}\} \\
=\ & \mathrm{Min}\{\mathrm{rk}_t(h, f_\varepsilon f_\xi^{-1}),\ \mathrm{rk}_t(h, f_\zeta f_\xi^{-1})\} \\
\geq\ & \mathrm{Min}\{\mathrm{rk}_t(h, f_\Upsilon f_\varepsilon^{-1}),\ \mathrm{rk}_t(h, f_\Upsilon f_\varepsilon^{-1})\} \\
=\ & \mathrm{rk}_t(h, f_\Upsilon f_\varepsilon^{-1}).
\end{aligned}$$

So giving also clause (2) of 1.14. $\square$

$\square_{1.1}$

Remark 1.20. The result about the cardinality of $Ext_p(G, \mathbf{Z})$ can be derived from Theorem 1.1 using the following definition (which constructs an isomorphic group ot $Ext_p(G, \mathbf{Z})$).

Definition 1.21. Given an abelian group G, let $G^* := Hom(G, \mathbf{Z})$ and for a prime p denote by G^p the group $Hom(G, \mathbf{Z}/p\mathbf{Z})$. For $g \in G^*$ let $g \mapsto g/p$ be the natural homomorphism from G^* into G^p. By G^*/p denote the subgroup of G^p which is the image ofG^*/p under $g \mapsto g/p$. Finally

$$Ext_p(G, \mathbf{Z}) := G^p/(G^*/p).$$

Recall that when λ is $\aleph_0$ or strong limit of cofinality $\aleph_0$ then $\lambda^{\aleph_0} = 2^\lambda$.

The group H_ω corresponde to the subgroup G^*/p and the σ's are inclusions.

We have learned from Paul Eklof that Christian U. Jensen in his book [Jen] have a proof of Theorem 1.0 of [GrSh] for the case that $\lambda = \aleph_0$.

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